Title:

Which Path is Fastest?

Solution:

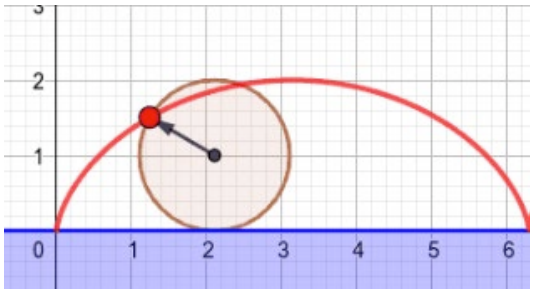
Johann Bernoulli posed the following problem in June, 1696. He said,

> *Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.*

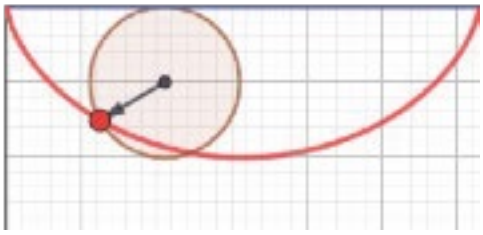
In other words, relying only on gravity, what path would allow a frictionless object to travel from point A to point B in the shortest time. We know the shortest distance is a straight line but the path that yields the shortest time is not.

Apparently it took Bernoulli two weeks to solve the problem but it took Newton less than 24 hours. Bernoulli allowed 6 months for people to submit their solutions. In the end, five mathematicians responded with solutions: Newton, Jakob Bernoulli (Johann's brother), Gottfried Leibniz, Ehrenfried Walther von Tschirnhaus and Guillaume de l'Hôpital. All of the solutions were published in the same journal where Bernoulli originally posed the problem - *Acta Eruditorum.*

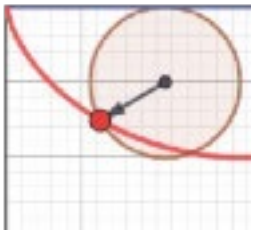
Interestingly the shape of the curve that yields the shortest time can be constructed using a cycloid. A cycloid is the curve found by placing a point on a circle and tracing the path of the point as the circle rolls (see the curve shown below).



If a cycloid is reflected vertically (see image A below) and then truncated to the halfway point (see image B below), the resulting shape is a brachistochrone – the path that yields the shortest travel time.



(A)



(B)

The parametric equations that can be used to generate a brachistochrone for a circle of radius 1 are

$$x(t) = t - \sin(t)$$

$$y(t) = -(1 - \cos(t))$$

The resulting graph of these parametric equations for the interval on t of $[0, \pi]$ is shown below: